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On a class of analytic functions related with generalized Bazilevic type functions

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ABSTRACT

The aim of this paper is to define and study a class of analytic functions related with generalized Bazilevic type functions. A necessary condition, arc length and coefficient difference are the main problems which we discussed here for this class.

1. Introduction

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Also let $S^*(\rho)$, $C(\rho)$ and $K(\rho)$ be the classes of starlike, convex and close-to-convex functions of order ρ $(0 \le \rho < 1)$ respectively, see [1].

Let $P_k^\alpha(\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and satisfying the condition

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} e^{i\alpha} p(z) - \rho \cos\alpha}{1-\rho} \right| d\theta \le k\pi \cos\alpha,$$

where $z = re^{i\theta}$, $k \ge 2$ and $0 \le \rho < 1$, α is real with $|\alpha| < \frac{\pi}{2}$. For $\alpha = 0$ and $\rho = 0$, we obtain the class P_k defined in [2] and for $k = 2$, we have the class P of functions with a positive real part.

Let $V_k^\alpha(\rho)$ be the class of functions $f(z)$ analytic in E and of the form (1.1) if and only if

$$\frac{(zf'(z))'}{f'(z)} \in P_k^\alpha(\rho), \quad z \in E.$$

When $\alpha = 0$, and $\rho = 0$, we have the class V_k of functions with bounded boundary rotation. The class $V_k^\alpha(\rho)$ was introduced and studied in [3].

It can easily be shown that $f(z) \in V_k^\alpha(\rho)$, if and only if, there exists $f_1(z) \in V_k$ such that

$$f'(z) = \left(f_1'(z)\right)^{(1-\rho)e^{-i\alpha}\cos\alpha}. \tag{1.2}$$

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Definition 1.1 ([4]). A function $f(z)$ analytic in E and of the form (1.1) belongs to the class $T_k^{\alpha}(\rho, b)$, $b \in \mathbb{C} - \{0\}$, $k \geq 2$, α real, $|\alpha| < \frac{\pi}{2}$ and $0 \leq \rho < 1$, if there exists a function $g(z) \in V_k^{\alpha}(\rho)$ such that

$$\frac{f'(z)}{g'(z)} \in P(b), \quad z \in E,$$

where $P(b)$ is the extension of the class P.

Definition 1.2 ([4]). A function $f(z)$ analytic in E and of the form (1.1) belongs to the class $Q_k^{\alpha}(a, \gamma, \rho, b)$ with $\frac{f(z)(f(z))'}{z} \neq 0$, $\operatorname{Re}\{a\} \geq 0$, $0 < \gamma \leq 1$, if there exists a function $g(z) \in T_k^{\alpha}(\rho, b)$ such that

$$zf'(z) + af(z) = (a+1)\, z \left(g'(z)\right)^{\gamma}, \quad z \in E.$$

The class of Bazilevic functions in the open unit disc E was first introduced by Bazilevic [5] in 1955. He defined Bazilevic function by the relation

$$f(z) = \left\{ \frac{\eta}{1+\beta^2} \int_0^z (p(t) - \beta \mathrm{i})\, t^{\frac{-\eta\beta \mathrm{i}}{2} - 1} g^{\frac{\eta}{1+\beta^2}}(t) \mathrm{d}t \right\}^{\frac{1+\beta^2}{\eta}},$$

where $p(z) \in P$, $g(z) \in S^*$, β is real and $\eta > 0$. For $\beta = 0$, we have the class of Bazilevic functions of type η and is given by

$$f(z) = \left\{ \eta \int_0^z p(t)\, t^{-1} g^{\eta}(t) \mathrm{d}t \right\}^{\frac{1}{\eta}}.$$

Using the above concepts we generalize the concept of Bazilevic functions of type η. We define the class $B_k^{\alpha}(\eta, \gamma, \rho, b)$ as follows:

Definition 1.3. A function $f(z) \in B_k^{\alpha}(\eta, \gamma, \rho, b)$, if and only if, there exist $g(z) \in Q_k^{\alpha}(0, \gamma, \rho, b)$ such that

$$\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{zg'(z)} \right)^{\eta} \in P_k, \quad z \in E, \tag{1.3}$$

where $k \geq 2$, $\eta > 0$, $0 \leq \rho < 1$, $0 < \gamma \leq 1$, $b \in \mathbb{C} - \{0\}$ and α real, $|\alpha| < \frac{\pi}{2}$.

By giving specific values to k, ρ, α, η, b and γ in $B_k^{\alpha}(\eta, \gamma, \rho, b)$, we obtain many important subclasses studied by various authors in earlier papers, see for details [6–11].

Throughout this article we assume that $k \geq 2$, $\eta > 0$, $0 \leq \rho < 1$, $0 < \gamma \leq 1$, $b \in \mathbb{C} - \{0\}$ and α real with $|\alpha| < \frac{\pi}{2}$ unless otherwise mentioned.

2. Preliminary lemmas

Lemma 2.1 ([6]). *Let $p(z)$ be analytic in E with $p(0) = 1$ belonging to the class P_k. Then for $z = r\mathrm{e}^{\mathrm{i}\theta}$*

$$\frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 \mathrm{d}\theta \leq \frac{1 + \left(k^2 - 1\right) r^2}{1 - r^2}.$$

Lemma 2.2 ([4]). *If $h(z)$ is analytic with $h(0) = 1$ and $h(z) \in P(b)$, then*

$$\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 \mathrm{d}\theta \leq \frac{1 + \left\{4|b|^2 - 1\right\} r^2}{1 - r^2}, \quad \text{for } z = r\mathrm{e}^{\mathrm{i}\theta}.$$

Lemma 2.3 ([12]). *Let $p_1(z) \in P$ and $z = r\mathrm{e}^{\mathrm{i}\theta}$. Then*

$$\int_0^{2\pi} |p_1(z)|^{\lambda} \mathrm{d}\theta \leq c(\lambda) \frac{1}{(1-r)^{\lambda - 1}},$$

where $\lambda > 1$ and $c(\lambda)$ is constant depending only on λ.

Lemma 2.4. *Let $s_1(z)$ be α-spirallike univalent function in E. Then*

(i) *there exists a ξ with $|\xi| = r$ such that for all z, $|z| = r$*

$$|z - \xi|\, |s_1(z)| \leq \frac{2r^2}{1 - r^2}, \quad \text{see [13]},$$

(ii)

$$\frac{r}{(1+r)^{2\cos\alpha}} \leq |s_1(z)| \leq \frac{r}{(1-r)^{2\cos\alpha}}, \quad \text{see [14]}.$$

Lemma 2.5 *([4]). Let $f(z) \in Q_k^{\alpha}(0, \gamma, \rho, b)$. Then*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} \mathrm{d}\theta > -\left[|b| + (1-\rho)\left(\frac{k}{2}-1\right)\cos^2\theta\right]\gamma\pi,$$

where $z = re^{i\theta}$ and $0 \leq \theta_1 < \theta_2 \leq 2\pi$.

3. Main results

Theorem 3.1. *A function $f(z) \in B_k^{\alpha}(\eta, \gamma, \rho, b)$, if and only if,*

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J(\eta, f(z))\, \mathrm{d}\theta > -\left[\frac{k}{2} + \gamma\eta\left\{|b| + \left(\frac{k}{2}-1\right)(1-\rho)\cos^2\theta\right\}\right]\pi, \tag{3.1}$$

where $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$, $r < 1$, and

$$J(\eta, f(z)) = \left\{1 + \frac{zf''(z)}{f'(z)}\right\} + (\eta - 1)\left\{\frac{zf'(z)}{f(z)}\right\}. \tag{3.2}$$

Proof. We can define, for $z = re^{i\theta}$, $r \in (0, 1)$, θ real, the following

$$S(r, \theta) = \arg\left[zf'(z)f^{\eta-1}(z)\right], \tag{3.3}$$

and

$$V(r, \theta) = \arg\left[(zg'(z))^{\eta}\right]. \tag{3.4}$$

The functions $S(z)$, $V(z)$ are periodic and continuous with period 2π. Since $f(z) \in B_k^{\alpha}(\eta, \gamma, \rho, b)$, therefore from [6], it follows that we can choose the branches of argument of $S(z)$ and $V(z)$ as

$$|S(r, \theta) - V(r, \theta)| < \frac{k\pi}{4}. \tag{3.5}$$

Now we have from (3.4)

$$V(r, \theta_2) - V(r, \theta_1) = \eta \int_{\theta_1}^{\theta_2} \left\{\frac{(zg'(z))'}{g'(z)}\right\} \mathrm{d}\theta.$$

Since $g(z) \in Q_k^{\alpha}(0, \gamma, \rho, b)$, therefore by using Lemma 2.5 and from (3.3)–(3.5), we have

$$\begin{aligned}|S(r, \theta_1) - S(r, \theta_2)| &= |S(r, \theta_1) + V(r, \theta_1) - V(r, \theta_1) + V(r, \theta_2) - V(r, \theta_2) - S(r, \theta_1)| \\ &< \left[\frac{k}{2} + \gamma\eta\left\{|b| + \left(\frac{k}{2}-1\right)(1-\rho)\cos^2\theta\right\}\right]\pi.\end{aligned}$$

Moreover, from (3.3)

$$\frac{\mathrm{d}}{\mathrm{d}\theta} S(r, \theta) = \operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)} + (\eta - 1)\frac{zf'(z)}{f(z)}\right\}.$$

Therefore

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} J(a, \eta, f(z))\, \mathrm{d}\theta > -\left[\frac{k}{2} + \gamma\eta\left\{|b| + \left(\frac{k}{2}-1\right)(1-\rho)\cos^2\theta\right\}\right]\pi. \quad \square$$

Theorem 3.2. *A function $f(z) \in B_k^{\alpha}(\eta, \gamma, \rho, b)$, if and only if,*

$$f(z) = \left\{\eta \int_0^z t^{\eta-1}\left(\frac{G(t)}{t}\right)^{\eta\gamma(1-\rho)e^{-i\alpha}\cos\alpha} (h(t))^{\eta\gamma}\, p(t)\mathrm{d}t\right\}^{\frac{1}{\eta}}, \tag{3.6}$$

where $h(z) \in P(b)$, $p(z) \in P_k$ and $G(z) \in R_k$, the class of bounded radius rotations studied by Tammi [15].

Proof. From (1.3), we have

$$z^{1-\eta}f'(z)(f)^{\eta-1}(z)=\left(g'(z)\right)^{\eta}p(z),$$

where $g(z)\in Q_k^{\alpha}(0,\gamma,\rho,b)$ and $p(z)\in P_k$. Now using the Definition 1.2, $g(z)\in Q_k^{\alpha}(0,\gamma,\rho,b)$, if and only if

$$g'(z)=\left(g_1'(z)\right)^{\gamma},\quad z\in E,$$

where $g_1(z)\in T_k^{\alpha}(\rho,b)$. This implies that

$$z^{1-\eta}f'(z)\left(f(z)\right)^{\eta-1}=\left(g_1'(z)\right)^{\gamma\eta}p(z).$$

Using Definition 1.1 along with (1.2), we have

$$z^{1-\eta}f'(z)\left(f(z)\right)^{\eta-1}=\left[\left(g_2'(z)\right)^{(1-\rho)\mathrm{e}^{-\mathrm{i}\alpha}\cos\alpha}h(z)\right]^{\gamma\eta}p(z),$$

where $g_2(z)\in V_k$ and $h(z)\in P(b)$. By an Alexander type relation we have for $G(z)\in R_k$

$$\eta f'(z)\left(f(z)\right)^{\eta-1}=\eta z^{\eta-1}\left(\frac{G(z)}{z}\right)^{\gamma\eta(1-\rho)\mathrm{e}^{-\mathrm{i}\alpha}\cos\alpha}\left(h(z)\right)^{\gamma\eta}p(z).$$

Integrating from 0 to z, we obtain the required result. □

Theorem 3.3. *Let $f(z)\in B_k^{\alpha}(\eta,\gamma,\rho,b)$ and $M(r)=\max_{|z|=r}|f(z)|$. Then*

$$M^{\eta}(r)\le\frac{2^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos\alpha+\gamma\eta-1}|b|^{\gamma\eta}(2+k)\,r^{\eta}}{\left[{}_2F_1\left(\eta;\gamma\eta\left\{(1-\rho)\left(\frac{k}{2}+1\right)\cos\alpha+1\right\}+1;\eta+1;r\right)\right]^{-1}},\tag{3.7}$$

where ${}_2F_1$ is the hypergeometric function.

Proof. From the Theorem 3.2, we have

$$f^{\eta}(z)=\left\{\eta\int_0^z t^{\eta-1}\left(G_1'(t)h(t)\right)^{\gamma\eta}p(t)\mathrm{d}t\right\},\quad z\in E,$$

where $G(z)\in V_k^{\alpha}(\rho)$, $h(z)\in P(b)$ and $p(z)\in P_k$. It is well known [16] that for α-spirallike functions $s_1(z)$ and $s_2(z)$

$$G'(z)=\frac{\left(\frac{s_1(z)}{z}\right)^{\left(\frac{k}{4}+\frac{1}{2}\right)(1-\rho)}}{\left(\frac{s_2(z)}{z}\right)^{\left(\frac{k}{4}-\frac{1}{2}\right)(1-\rho)}}.\tag{3.8}$$

Therefore

$$f^{\eta}(z)=\eta\int_0^z t^{\eta-1}\left[\frac{\left(\frac{s_1(t)}{t}\right)^{\left(\frac{k}{4}+\frac{1}{2}\right)(1-\rho)}}{\left(\frac{s_2(t)}{t}\right)^{\left(\frac{k}{4}-\frac{1}{2}\right)(1-\rho)}}\right]^{\gamma\eta}h^{\gamma\eta}(t)p(t)\,\mathrm{d}t.$$

Using the Lemma 2.4 and well-known distortion results for the classes $P(b)$ and P_k, we obtain

$$|f^{\eta}(z)|\le\eta2^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos\alpha-1}(2|b|)^{\gamma\eta}(2+k)\int_0^r\frac{t^{\eta-1}\mathrm{d}t}{(1-t)^{\gamma\eta\left\{(1-\rho)\left(\frac{k}{2}+1\right)\cos\alpha+1\right\}+1}}.$$

Now for $t=ru$, we obtain

$$\begin{aligned}M(r)^{\eta}&\le\eta2^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos\alpha-1}(2|b|)^{\gamma\eta}(2+k)\,r^{\eta}\int_0^1\frac{u^{\eta-1}\mathrm{d}u}{(1-ru)^{\gamma\eta\left\{(1-\rho)\left(\frac{k}{2}+1\right)\cos\alpha+1\right\}+1}}\\&=\frac{2^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos\alpha+\gamma\eta-1}|b|^{\gamma\eta}(2+k)\,r^{\eta}}{\left[{}_2F_1\left(\eta;\gamma\eta\left\{(1-\rho)\left(\frac{k}{2}+1\right)\cos\alpha+1\right\}+1;\eta+1;r\right)\right]^{-1}}.\end{aligned}$$

Hence the proof is complete. □

Theorem 3.4. *Let $f(z) \in B_k^\alpha(\eta, \gamma, \rho, b)$, $0 < \eta \le 1, 0 < \gamma < 1$ and $\lambda > 1$. Then, for $M(r) = \max_{|z|=r} |f(z)|$ and $r \to 1$*

$$L_r f(z) \le C(\alpha, \eta, \gamma, \rho, b, k) M^{1-\eta}(r) \left(\frac{1}{1-r}\right)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]}, \tag{3.9}$$

where $L_r f(z)$ denotes the arc length and $C(\alpha, \eta, \gamma, \rho, b, k)$ is a constant depending upon $\alpha, \eta, \gamma, \rho, b$, and k only.

Proof. We know that, for $z = re^{i\theta}$, $0 < r < 1$, $0 \le \theta \le 2\pi$

$$L_r f(z) = \int_0^{2\pi} |zf'(z)| d\theta.$$

Now from Theorem 3.2, we have

$$zf'(z) = z^\eta f^{1-\eta}(z) \left(g'(z)\right)^\eta p(z),$$

where $g(z) \in Q_k^\alpha(0, \gamma, \rho, b)$ and $p(z) \in P_k$. Since $g(z) \in Q_k^\alpha(0, \gamma, \rho, b)$, therefore we have

$$g'(z) = \left(g_1'(z) h(z)\right)^\gamma,$$

where $g_1(z) \in V_k^\alpha(\rho)$ and $h(z) \in P$. This implies that from (1.2)

$$L_r f(z) = \int_0^{2\pi} \left| z^\eta f^{1-\eta}(z) \left(\left(g_2'(z)\right)^{(1-\rho)e^{-i\alpha}\cos\alpha} h(z)\right)^{\gamma\eta} p(z) \right| d\theta, \tag{3.10}$$

where $h(z) \in P(b)$, $p(z) \in P_k$ and $g_2(z) \in V_k$.

Now, it is known [17] that for $g_2(z) \in V_k$, there exists a starlike function $s(z)$ and $h_1(z) \in P$ such that

$$zg_2'(z) = s(z)\ (h_1(z))^{\frac{k}{2}-1}, \quad z \in E. \tag{3.11}$$

Also in [18], it is given that $s_1(z)$ is α-spirallike function, if and only if, there is a starlike function $s(z)$ such that

$$s_1(z) = z\left[\frac{s(z)}{z}\right]^{e^{-i\alpha}\cos\alpha}, \quad z \in E. \tag{3.12}$$

From (3.10)–(3.12), we obtain

$$L_r f(z) = \int_0^{2\pi} \left| z^\eta f^{1-\eta}(z) \left(\frac{s_1(z)}{z}\right)^{\gamma\eta(1-\rho)} (h_1(z))^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)e^{-i\alpha}\cos\alpha} h^{\gamma\eta}(z) p(z) \right| d\theta, \tag{3.13}$$

where $h_1(z) \in P$, $h(z) \in P(b)$, $p(z) \in P_k$ and $s_1(z)$ is an α-spirallike function. This implies that from (3.13)

$$L_r f(z) \le 2\pi r^{\eta+\gamma\eta(1-\rho)} M^{1-n}(r) e^{\pi\frac{\sin 2\alpha}{4}} \frac{1}{2\pi}\int_0^{2\pi} |(s_1(z))|^{\gamma\eta(1-\rho)} |h(z)|^{\gamma\eta} |p(z)|\, |(h_1(z))|^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos^2\alpha} d\theta.$$

Now using the generalized Holder inequality for $\lambda_1 = 2$, $\lambda_2 = \frac{2}{\gamma\eta}$, $\lambda_3 = \frac{4}{1-\gamma\eta}$, $\lambda_4 = \frac{4}{1-\gamma\eta}$ are all positive for $0 < \eta \le 1$, $0 < \gamma < 1$ such that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} = 1$. Therefore, we have

$$L_r f(z) \le 2\pi r^{\eta+\gamma\eta(\rho-1)} M^{1-\eta}(r) e^{\pi\frac{\sin 2\alpha}{4}} \left(\frac{1}{2\pi}\int_0^{2\pi} |(s_1(z))|^{\frac{4\gamma\eta(1-\rho)}{1-\gamma\eta}} d\theta\right)^{\frac{1-\gamma\eta}{4}} \left(\frac{1}{2\pi}\int_0^{2\pi} |p(z)|^2 d\theta\right)^{\frac{1}{2}}$$
$$\times \left(\frac{1}{2\pi}\int_0^{2\pi} |(h_1(z))|^{\frac{4\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos^2\alpha}{1-\gamma\eta}} d\theta\right)^{\frac{1-\gamma\eta}{4}} \left(\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{\gamma\eta}{2}}.$$

By Lemmas 2.1–2.4 and a subordination result, we have

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |(s_1(z))|^{\frac{4\gamma\eta(1-\rho)}{1-\gamma\eta}} d\theta\right)^{\frac{1-\gamma\eta}{4}} \le \left(\frac{1}{2\pi}\right)^{\frac{1-\gamma\eta}{4}} \left(\frac{1}{1-r}\right)^{2\gamma\eta(1-\rho)\cos\alpha - \frac{1-\gamma\eta}{4}}, \quad r < 1,$$

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |h(z)|^2 d\theta\right)^{\frac{\gamma\eta}{2}} \le \left(\frac{1+\left\{4|b|^2-1\right\}r^2}{1-r^2}\right)^{\frac{\gamma\eta}{2}} \le 2^{\frac{\gamma\eta}{2}} |b|^{\gamma\eta} \left(\frac{1}{1-r}\right)^{\frac{\gamma\eta}{2}},$$

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2 d\theta\right)^{\frac{1}{2}} \leq \left(\frac{1+\left(k^2-1\right)r^2}{1-r^2}\right)^{\frac{1}{2}} \leq 2^{-\frac{1}{2}}k\left(\frac{1}{1-r}\right)^{\frac{1}{2}},$$

$$\left(\frac{1}{2\pi}\int_0^{2\pi}|(h_1(z))|^{\frac{4\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos^2\alpha}{1-\gamma\eta}}d\theta\right)^{\frac{1-\gamma\eta}{4}} \leq (c(\lambda))^{\frac{1-\gamma\eta}{4}}\left(\frac{1}{1-r}\right)^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos^2\alpha-\frac{1-\gamma\eta}{4}},$$

with $\lambda = \frac{4\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)}{1-\gamma\eta}\cos^2\alpha > 1$. Therefore, we have

$$\begin{aligned} L_r f(z) &\leq C(\alpha,\eta,\gamma,\rho,b,k)\,M^{1-\eta}(r)\left(\frac{1}{1-r}\right)^{2\gamma\eta(1-\rho)\cos\alpha-\frac{1-\gamma\eta}{4}+\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)\cos^2\alpha-\frac{1-\gamma\eta}{4}+\frac{\gamma\eta}{2}+\frac{1}{2}} \\ &= C(\alpha,\eta,\gamma,\rho,b,k)\,M^{1-\eta}(r)\left(\frac{1}{1-r}\right)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]}, \quad r\to 1. \quad \square \end{aligned}$$

Theorem 3.5. *Let $f(z)\in B_k^\alpha(\eta,\gamma,\rho,b)$, $0<\eta\leq 1$, $0<\gamma<1$ and $\lambda>1$. Then for $n\geq 2$*

$$|a_n| \leq C_1(\alpha,\eta,\gamma,\rho,b,k)\,M^{1-\eta}(n)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]-1}, \tag{3.14}$$

where $C_1(\alpha,\eta,\gamma,\rho,b,k)$ is constant depending upon α, η, γ, ρ, b and k only.

Proof. Since with $z=re^{i\theta}$, the Cauchy theorem gives

$$na_n = \frac{1}{2\pi r^n}L_r f(z).$$

Using Theorem 3.4, we obtain

$$n|a_n| \leq \frac{1}{2\pi r^n}C(\alpha,\eta,\gamma,\rho,b,k)\,M^{1-\eta}(r)\left(\frac{1}{1-r}\right)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]}.$$

Taking $r=1-\frac{1}{n}$, we have

$$|a_n| \leq C_1(\alpha,\eta,\gamma,\rho,b,k)\,M^{1-\eta}(n)(n)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]-1}, \quad n\to\infty$$

which is the required result. $\square$

Theorem 3.6. *Let $f(z)\in B_k^\alpha(\eta,\gamma,\rho,b)$, $0<\eta\leq 1$, $0<\gamma<1$ and $\lambda>1$. Then for $n\geq 2$*

$$\left||a_{n+1}|-|a_n|\right| \leq C_2(\alpha,\eta,\gamma,\rho,b,k)\,M^{1-\eta}(n)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]-2\cos\alpha-2}, \tag{3.15}$$

where $C_2(\alpha,\eta,\gamma,\rho,b,k)$ is a constant depending upon α, η, γ, ρ, b and k only.

Proof. We know that

$$|(n+1)\xi a_{n+1}-na_n| \leq \frac{1}{2\pi r^n}\int_0^{2\pi}|z-\xi|\,|zf'(z)|d\theta.$$

Since

$$zf'(z) = z^\eta f^{1-\eta}(z)\left(\frac{s_1(z)}{z}\right)^{\gamma\eta(1-\rho)}(h_1(z))^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)e^{-i\alpha}\cos\alpha}h^{\gamma\eta}(z)\,p(z),$$

where $h_1(z)\in P$, $h(z)\in P(b)$, $p(z)\in P_k$ and $s_1(z)$ is α-spirallike function, therefore

$$\begin{aligned} |(n+1)\xi a_{n+1}-na_n| &\leq \frac{1}{2\pi r^n}\int_0^{2\pi}|z-\xi|\left|\frac{z^\eta f^{1-\eta}(z)\left(\frac{s_1(z)}{z}\right)^{\gamma\eta(1-\rho)}h^{\gamma\eta}(z)p(z)}{(h_1(z))^{-\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)e^{-i\alpha}\cos\alpha}}\right|d\theta, \\ &\leq e^{\frac{\pi\sin 2\alpha}{4}}M^{1-\eta}(r)r^{\eta-\gamma\eta(1-\rho)}\frac{1}{2\pi r^n}\int_0^{2\pi}\left\{|z-\xi|\,|s_1(z)|\,|s_1(z)|^{\gamma\eta(1-\rho)-1}|h_1(z)|^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)e^{-i\alpha}\cos\alpha}|h(z)|^{\gamma\eta}|p(z)|d\theta\right\} \\ &= e^{\frac{\pi\sin 2\alpha}{4}}\frac{1}{2\pi r^n}M^{1-\eta}(r)r^{\eta-\gamma\eta(1-\rho)}\frac{2r^2}{1-r^2}\int_0^{2\pi}|s_1(z)|^{\gamma\eta(1-\rho)-1}|h_1(z)|^{\gamma\eta(1-\rho)\left(\frac{k}{2}-1\right)e^{-i\alpha}\cos\alpha}|h(z)|^{\gamma\eta}|p(z)|d\theta. \end{aligned}$$

Now by using a similar procedure as given in Theorem 3.4, we obtain

$$|(n+1)\xi a_{n+1} - na_n| \le C_2(\alpha, \eta, \gamma, \rho, b, k) M^{1-\eta}(r) r^{\eta-\gamma\eta(1-\rho)-n} \left(\frac{1}{1-r}\right)^{\gamma\eta\left[(1-\rho)\cos\alpha\left(\left(\frac{k}{2}-1\right)\cos\alpha+2\right)+1\right]-2\cos\alpha-1}.$$

Putting $|\xi| = r = \frac{n}{1+n}$, we have the required result. $\square$

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